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Alhazen's Problem.

Its Bibliography and an Extension of the Problem.

By Marcus Baker.

PROBLEM. *From two points in the plane of a circle, to draw lines meeting at a point in the circumference and making equal angles with the tangent drawn at that point.*

This problem is known as Alhazen's, and has been studied by several mathematicians, besides Alhazen, from the time of Huyghens to the present. The earliest solutions are all geometrical constructions in which the points are determined by the intersections of a hyperbola with the given circle. Later, analytical solutions were given, and lastly trigonometrical solutions.

The following list of references and their accompanying notes contains a condensed history of the problem.

ALHAZEN. Opticae thesavrvs Alhazeni arabis libri septem nunc primum editi . . . à Risnero. 4 p.l. 288 pp. fol. *Basileae per episcopios* MDLXXII.

Alhazen was an Arabian who lived in the 11th century of the Christian era, dying at Cairo in 1038. He wrote this treatise on optics, which was first published, as above stated, in 1572, under the editorship of Risner. The first published solution of the problem is contained in this book, pp. 144–148, Props. 34, 38 and 39. The solution is effected by the aid of a hyperbola intersecting a circle, and is excessively prolix and intricate.

ANALYST (The). A journal of pure and applied mathematics. Edited and published by J. E. Hendricks, A. M. 8vo. *Des Moines, Iowa: Mills & Co.* 1877. Vol. IV, No. 3, pp. 124–125.

A general solution was proposed, but the solution of only a special case was published.

BARROW (*Rev.* Isaac, *D. D.*) Lectiones xviii Cantabrigiae in scholis publicis habitae in qvibvs opticorum phaenomenΩn genuinae rationes investigantur, ac exponuntur. 8vo. *Londini, typis Gulielmi Godbid,* 1669. Lect. ix.

See also WHEWELL's edition of Barrow's works. 8vo. *Cambridge, University Press*, 1860. Lectiones opticae, pp. 84–89.

Barrow refers to Alhazen's "horribly prolix" solution, and discusses some of its special cases as applied to optics.

HUTTON (*Dr.* Charles). Trigonometrical solution of Alhazen's problem.

[*In* LEYBOURN (Thomas). The mathematical questions proposed in the Ladies' Diary (etc.) 8vo. *London: J. Mawman*, 1817. Vol. I, pp. 167–168.]

A trigonometrical solution by trial and error.

HUYGHENS (Christian) and SLUSE (Francis Rene). Excerpta ex epistolis nonnullis (etc.)

[*In* PHILOSOPHICAL TRANSACTIONS of the Royal Society. 4to. *London*, 1673. Vol. VIII, pp. 6119–6126, 6140–6146.]

Several solutions are contained in this correspondence, all of them geometrical, and all except one being by the aid of an hyperbola. One solution, by Sluse, is by means of the intersection of a parabola with the given circle. One of the solutions by Huyghens is the most elegant the problem has ever received.

KAESTNER (Abraham Gotthelf). Problematis Alhazeni analysis trigonometrica.

[*In* NOVI COMMENTARII societatis regiae scientiarum GOTTINGENSIS tomus VII, 1776. 4to. *Gottingae, J. C. Dieterich*, 1777, pp. 92–141. 1 pl.]

A complete trigonometrical solution, with application to several numerical examples, and containing differential equations to facilitate the computation.

LADIES' DIARY. 16mo. *London*, 1727. *Not seen.*

The problem concretely stated was the prize problem in the Diary for 1727. Solved the following year by trial and error by I . . . T . . .

LEYBOURN (Thomas). Geometrical construction of Alhazen's problem.

[*In* LEYBOURN (Thomas). The mathematical questions proposed in the Ladies' Diary (etc.) 8vo. *London: J. Mawman*, 1817. Vol. I, pp. 168–169.]

This construction is arranged from old material. The proof is given clearly and concisely, and a few bibliographic indications are added.

L'HOSPITAL (Guillaume François Antoine de). Traité analytique de sections coniques. 4to. *Paris: Montalant*, 1720. Livre X, Ex. vii, pp. 389–395.

In Book X, on determinate sections, Alhazen's problem is selected as an example of a problem whose geometrical solution can be effected by the aid of a conic section. Two solutions are given, essentially the same as those by Huyghens and Sluse, but with improved methods of arrangement and proof.

MAYER (Tobias). In a collection of problems by Mayer there is said to be a solution of Alhazen's problem. *Not seen.*

PHILOSOPHICAL TRANSACTIONS. *See* HUYGHENS and WALES.

PRIESTLY (Joseph). History of optics, translated into German with notes by Simon Klügel. 2 vols. Leipzig, 1775–6. *Not seen.*

Said to contain information about Alhazen's problem.

RISNER (Frederic)—Editor. *See* ALHAZEN.

ROBINS (Benjamin). Mathematical tracts. 2 vols. 8vo. *London,* 1761. *See* Vol. II, pp. 262–264.

One of these tracts is a scathing review of Robert Smith's "compleat system of optics." Robins points out the complete omission of Alhazen's problem, and supplies a short and easy proof of the correctness of one of Sluse's solutions.

SEITZ (Enoch Beery). Solution of a problem.

[*In* SCHOOL (The) VISITOR, devoted to the study of mathematics and grammar. 8vo. *Ansonia, Ohio: John S. Royer,* 1881. Vol. II, No. 2, February, pp. 24–25.]

A complete algebraical solution by an equation of the eighth degree, with numerical application and roots found by Horner's method.

SIMSON (Robert). Said to have solved the problem, but no solution is contained in any of his works accessible to the author.

SLUSE (Francis Rene). *See* HUYGHENS and SLUSE.

T . . . (I . . .). *See* LADIES' DIARY.

WALES (William). On the resolution of adfected equations.

[*In* PHILOSOPHICAL TRANSACTIONS of the Royal Society. 4to. *London,* 1781. Vol. LXXI, part 1, Ex. vi, pp. 472–476.]

Alhazen's problem is selected as furnishing an example of an adfected quadratic equation whose solution is easily effected by the aid of the logarithmic tables and his method of using them.

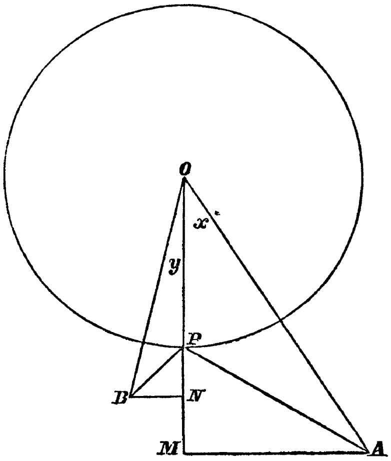
Alhazen's Problem extended to the Surface of a Sphere.

The solution of Alhazen's problem gives the minimum (and also maximum) path between two points and an intermediary circle, the points and circle being situated in the same plane, and we shall here give the solution of the same problem when the two points and circumference of the given circle are situated in the surface of a sphere.

To fix the ideas, consider the terrestrial spheroid to be a sphere, and let our given points be

$$A \text{ of which the latitude} = \phi_1 \text{ and longitude } \lambda_1,$$
$$\text{and } B \text{ " " "} = \phi_2 \text{ " " } \lambda_2.$$

The shortest distance between these points on the surface of the sphere is along the arc of the great circle joining them. Suppose this great circle track passes north of a given parallel of latitude ϕ; we wish to find the minimum path between A and B on the spherical surface which does *not* pass to the north of latitude ϕ.



The shortest path consists of two arcs of great circles drawn from A and B to a point P in latitude ϕ in such a manner as to make equal angles with the parallel ϕ at P. We proceed to determine this point P.

Referring to the annexed diagram, let us for a moment consider Alhazen's problem without the extension. Let $OA = a$, $OB = b$, and $OP = r$; then

$$\tan APM = \frac{a \sin x}{a \cos x - r} \text{ and } \tan BPN = \frac{b \sin y}{b \cos y - r},$$

and as these angles are equal we have

$$\frac{a \sin x\,(b \cos y - r)}{b \sin y\,(a \cos x - r)} = 1,$$

in which $x + y = \alpha$ a known angle. Replacing y by $\alpha - x$, we have a convenient formula for solving by approximation.

The solution for the extended problem is analogous to the foregoing. O is the pole of the sphere, A and B the given points, and P the point sought. OA, OB and OM are arcs of great circles, as are also PA and PB; and AM and BN are arcs of great circles perpendicular to OP. Let $AOM = x$, $BOM = y$, and $OP = 90^\circ - \phi$.

Then

$$\sin AM = \sin AO \sin x,$$
$$\sin BN = \sin BO \sin y,$$

and

$$\tan OM = \tan AO \cos x,$$
$$\tan ON = \tan BO \cos y;$$

also
$$\tan APM = \tan BPM \quad \text{or} \quad \frac{\tan AM}{\sin PM} = \frac{\tan BN}{\sin PN}$$

whence
$$\frac{\tan^2 AM}{\tan^2 BN} = \frac{\sin^2 PM}{\sin^2 PN}. \tag{1}$$

Now
$$\frac{\tan^2 AM}{\tan^2 BN} = \frac{\sin^2 AM \cos^2 BN}{\cos^2 AM \sin^2 BN} = \frac{\sin^2 AO \sin^2 x\,(1 - \sin^2 BO \sin^2 y)}{(1 - \sin^2 AO \sin^2 x) \sin^2 BO \sin^2 y}; \tag{2}$$

and again, since $PN = ON - OP$ and $PM = OM - OP$,

$$\frac{\sin^2 PM}{\sin^2 PN} = \left\{\frac{\sin OM \sin\varphi - \cos OM \cos\varphi}{\sin ON \sin\varphi - \cos ON \cos\varphi}\right\}^2 = \frac{\cos^2 OM}{\cos^2 ON}\left\{\frac{\tan OM \sin\varphi - \cos\varphi}{\tan ON \sin\varphi - \cos\varphi}\right\}^2.$$

But
$$\frac{\cos^2 OM}{\cos^2 ON} = \frac{1 + \tan^2 ON}{1 + \tan^2 OM} = \frac{1 + \tan^2 BO \cos^2 y}{1 + \tan^2 AO \cos^2 x},$$

whence
$$\frac{\sin^2 PM}{\sin^2 PN} = \left\{\frac{1 + \tan^2 BO \cos^2 y}{1 + \tan^2 AO \cos^2 x}\right\} \cdot \left\{\frac{\tan AO \cos x \sin\varphi - \cos\varphi}{\tan BO \cos y \sin\varphi - \cos\varphi}\right\}^2. \tag{3}$$

Substituting (2) and (3) in (1), and remembering that

$$AO = 90^\circ - \phi_1 \quad \text{and} \quad BO = 90^\circ - \phi_2$$

we obtain

$$\frac{\cos^2\varphi_1 \sin^2 x\,(1 - \cos^2\varphi_2 \sin^2 y)}{(1 - \cos^2\varphi_1 \sin^2 x)\cos^2\varphi_2 \sin^2 y} = \left\{\frac{1 + \operatorname{cotg}^2\varphi_2 \cos^2 y}{1 + \operatorname{cotg}^2\varphi_1 \cos^2 x}\right\} \cdot \left\{\frac{\operatorname{cotg}\varphi_1 \sin\varphi \cos x - \cos\varphi}{\operatorname{cotg}\varphi_2 \sin\varphi \cos y - \cos\varphi}\right\}^2,$$

or

$$\frac{\cos^2\varphi_1 \sin^2 x\,(1 - \cos^2\varphi_2 \sin^2 y)(1 + \operatorname{cotg}^2\varphi_1 \cos^2 x)(\operatorname{cotg}\varphi_2 \cos y - \operatorname{cotg}\varphi)^2}{\cos^2\varphi_2 \sin^2 y\,(1 - \cos^2\varphi_1 \sin^2 x)(1 + \operatorname{cotg}^2\varphi_2 \cos^2 y)(\operatorname{cotg}\varphi_1 \cos x - \operatorname{cotg}\varphi)^2} = 1.$$

Now $x + y = \lambda_2 - \lambda_1 = \Delta\lambda$, whence eliminating x or y we have an equation which can be solved by approximation as in the case of a plane surface.

WASHINGTON, D. C., *October* 26, 1881.